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Asymptotic Expansion of the Pressure in the Inverse Interaction Range

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We consider an Ising system in $d \geqslant 2$ dimensions with a ferromagnetic Kac potential whose scaling parameter is denoted by γ. We derive an asymptotic series for the thermodynamic pressure $P_{\beta, \gamma}$ in powers of γ. The 0th-order term of the expansion is the mean-field pressure of the Lebowitz and Penrose theory.

KEY WORDS: Ferromagnetic Kac potential; Ising system; pressure; correlation functions.

1. INTRODUCTION AND MAIN RESULTS

The approximation of mean field by long range interactions has a long history which goes back to the pioneering works of Kac, Uhlenbeck and Hemmer,[7] and Lebowitz and Penrose,[10] who have proved that the approximation becomes exact for Kac potentials in a scaling limit $\gamma \to 0$. γ^{-1} denotes the range of the Kac potential which is scaled so that the interaction of a molecule with all the others is bounded uniformly as $\gamma \to 0$ (the Lebowitz–Penrose limit).

The analsyis before the limit $\gamma \to 0$, when $\gamma > 0$ is kept fixed, is also very interesting. The interaction in this case has finite range and the system is a perfectly legitimate model of statistical mechanics. Yet, if γ is small, it is close to mean field and it can be studied using perturbative techniques w.r.t to the mean field behavior, with the inverse interaction range which plays the role of the inverse temperature β in Peierls estimates and Pirogov–Sinai methods. In this way, for instance, it is possible to study the

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"liquid-vapour branch" of the phase diagram by low temperature techniques, a strategy which has been succesful both for lattice and continuum systems.[5, 3, 9]

In the whole approach and particularly when applying the Pirogov–Sinai scheme, it is important to have an accurate control of the pressure for small values of γ. The γ-corrections to the mean field value of the pressure have been studied by J. Lebowitz, G. Stell, S. Baer, and W. Theumann in ref. 8, who have derived the first order terms of an asymptotic expansion of the pressure for values of the parameters where the Mayer and the low fugacity expansions apply.

In the [simpler] case of an Ising model in $d \geqslant 2$ dimensions with ferromagnetic Kac interactions, we have been able to extend their analysis going below the critical temperature, with phase transitions present. We derive in fact in this paper an asymptotic series for the pressure [and for the correlation functions as well] in powers of γ.

1.1. The Model

We study an Ising spin system on $\mathbb{Z}^d$. $\sigma = \{\sigma(x), x \in \mathbb{Z}^d\}$ is a spin configuration, $\sigma^\Lambda = \{\sigma(x), x \in \Lambda\}$ its restriction to a bounded region $\Lambda \subset \mathbb{Z}^d$. The energy of σ^Λ is

$$H_\gamma(\sigma^\Lambda) = -\frac{1}{2} \sum_{x \neq y \in \Lambda} J_\gamma(x, y)\, \sigma(x)\, \sigma(y) \tag{1.1}$$

where $\gamma > 0$ is the scaling parameter of the Kac potential $J_\gamma(x, y)$:

$$J_\gamma(x, y) := a_\gamma \gamma^d J(\gamma(x - y)), \qquad a_\gamma^{-1} = \sum_{x \neq 0} \gamma^d J(\gamma x) \tag{1.2}$$

(as in ref. 4, we normalize the interaction to have always, i.e. for all γ, total strength equal to 1). We suppose that $J(r)$, $r \in \mathbb{R}^d$ is a non-negative, smooth function supported by the unit ball and normalized so that

$$\int_{\mathbb{R}^d} dr\, J(r) = 1 \tag{1.3}$$

We have restricted to the case without magnetic field because we want to study phase transitions, our analysis, however, can be easily extended to non zero magnetic fields. We recall that the thermodynamic pressure is

$$P_\gamma(\beta) = \lim_{\Lambda \to \mathbb{Z}^d} \frac{1}{\beta\, |\Lambda|} \log Z_\gamma(\beta; \Lambda) \tag{1.4}$$

where β is the inverse temperature;

$$Z_\gamma(\beta; \Lambda) = \sum_{\sigma^\Lambda} e^{-\beta H_\gamma(\sigma^\Lambda)} \tag{1.5}$$

the partition function; the limit in (1.4) is (for instance) over an incresing sequence of cubes Λ.

In ref. 10 it is proved that

$$\lim_{\gamma \to 0} P_\gamma(\beta) = P^{\mathrm{mf}}(\beta) = - \lim_{m \in [-1, 1]} \left\{ -\frac{m^2}{2} - \frac{1}{\beta} I(m) \right\} \tag{1.6}$$

where $I(m)$ is the entropy at magnetization m

$$I(m) = -\frac{1-m}{2} \log \frac{1-m}{2} - \frac{1+m}{2} \log \frac{1+m}{2} \tag{1.7}$$

When $\beta \leqslant 1$, the minimizer is $m_\beta = 0$, while for $\beta > 1$ there are two minimizers, $\pm m_\beta$, m_β the positive root of the mean field equation

$$m_\beta = \tanh(\beta m_\beta) \tag{1.8}$$

Our main result is:

Theorem 1.1. For any $\beta \neq 1$ there are bounded functions $p_n(\beta, \gamma)$ which have non zero limit as $\gamma \to 0$ and such that

$$P_\gamma(\beta) \sim P^{\mathrm{mf}}(\beta) + \sum_{n \geqslant 1} \gamma^{nd} p_n(\beta, \gamma) \tag{1.9}$$

(i.e., the r.h.s. is an asymptotic series for the l.h.s. see (1.10) below).

As we shall see in the next section the coefficients $p_n(\beta, \gamma)$ have an "explicit" diagramatic representation as solutions of linear systems of equations (derived from the BBGKY hierarchy of equations for the correlation functions). They still depend on γ and they could be further expanded so that we would then get an asymptotic series in powers of γ. This is however not so meaningful physically, as it just amounts to expand in γ quantities like the moments of $J_\gamma(\cdot)$, and we have decided to state our result in the more compact form (1.9).

As mentioned the series (1.9) is an asymptotic series, namely for any $N \geqslant 1$

$$\lim_{\gamma \to 0} \gamma^{-Nd} \left(P_\gamma(\beta) - P^{\mathrm{mf}}(\beta) - \sum_{n=1}^{N} \gamma^{nd} p_n(\beta, \gamma) \right) = 0 \tag{1.10}$$

Theorem 1.1 neither claims the convergence of the series on the r.h.s. of (1.9) nor equality with the l.h.s. which we do not expect, at least for $\beta > 1$. In fact, by its definition (1.10), the asymptotic series neglects terms like $e^{-c\gamma^{-d}}$ which are certainly present as they are related to the probability of contours, which contribute to large deviations and surface tension.[1, 2] For the same reasons the pressure in "contour models" has the same asymptotic series given by (1.9).

The asymptotic series at $\beta < 1$ and $\beta > 1$ have similar structure and the coefficients vary continuously with β, but the continuation is not more regular than continuous. Our analysis does not apply at $\beta = 1$.

Finally a few comments about the proof, which is presented in the next section. We do not use cluster expansion techniques, which are certainly available at β small (uniformly in γ) and could be possibly proved also when there is a phase transition, by working with contour models. Our proof is much more elementary, as we first relate the pressure to the two body correlation functions of an interpolating hamiltonian and then work directly on the BBGKY hierarchy, in the same spirit as in ref. 8. Below the critical temperature we use a modified hierarchy which involves truncated correlations functions, the so called v-functions. This gives quite naturally and directly a formal expansion in powers of γ^d. To make it a true asymptotic expansion though, we need to control the remainder. This is done in two steps, we first prove that a n-body v-function can be expressed in terms of n-fold convolutions of the same v-functions with the interaction. We then estimate such smoothened expressions by Peierls estimates on [moderately] large deviations. All that becomes simpler at $\beta < 1$ where we can work directly with correlation functions and use Gaussian inequalities to reduce the bound on the n-body correlation functions to two body correlation functions, as in ref. 4.

2. PROOFS

As mentioned above, we will prove Theorem 1.1 by relating the pressure to the v-functions of an interpolating Gibbs measure, Subsection 2.1, and then studying a BBGKY like hierarchy for the v-functions.

2.1. Reduction to Pair Correlation Functions

When $\beta < 1$, following ref. 6 we interpolate with the free model at $\beta = 0$. Letting $t \in [0, 1]$ we have

$$P_\gamma(\beta) = \mathscr{P}^0(\beta) + \frac{1}{2} \sum_{y \neq 0} J_\gamma(0, y) \int_0^1 dt\, E_{\mu_{\gamma, t\beta}}(\sigma(0)\, \sigma(y)), \qquad \mathscr{P}^0(\beta) = \frac{1}{\beta} \log 2 \tag{2.1}$$

The analogue of (2.1) obviously holds in a finite region Λ, (2.1) is then obtained by letting $\Lambda \to \mathbb{Z}^d$. In fact the finite volume correlation functions converge to their infinite volume limit because the Dobrushin uniqueness condition holds for any $\beta < 1$ and all γ, see for instance ref. 4.

The asymptotic expansion of $P_\gamma(\beta)$ is thus determined, via (2.1), by the asymptotic expansion of the two body correlation functions. The analogous expression when $\beta > 1$ is not as useful, because, in such a case $t\beta$ goes through the critical temperature as $t \in [0, 1]$. However, since for γ small enough, any extremal Gibbs measure is close to a Bernoulli process with non zero magnetization,[3] it is then more convenient to interpolate with such a free measure. We recall from ref. 3 that when $\beta > 1$ and γ small there are two and only two translationally invariant Gibbs states, $\mu^{\pm}_{\gamma,\beta}$ obtained by taking the thermodynamic limit with $+$ and $-$ boundary conditions. For sake of definitiness, in the sequel we restrict to $\beta > 1$.

Theorem 2.1. For any $\beta > 1$ there are $c > 0$ and c' positive so that the following holds. Let Λ be a cube, $\mu^{\Lambda,+}_{\gamma,\beta}$ the Gibbs measure on $\{-1, 1\}^\Lambda$

$$\mu^{\Lambda,+}_{\gamma,\beta}(\sigma^\Lambda) = \frac{e^{-\beta H^+_\gamma(\sigma^\Lambda)}}{Z^+_\gamma(\beta; \Lambda)}, \qquad H^+_\gamma(\sigma^\Lambda) = H_\gamma(\sigma^\Lambda) - \sum_{x \in \Lambda,\, y \notin \Lambda} J_\gamma(x, y)\, \sigma(x)\, m_\beta \tag{2.2}$$

with $Z^+_\gamma(\beta; \Lambda)$ the normalizing factor. Let $\Delta \subset \Lambda$ and f any function of σ^Δ, then

$$|E_{\mu^{\Lambda,+}_{\gamma,\beta}}(f) - E_{\mu^+_{\gamma,\beta}}(f)| \leqslant c' e^{-c\gamma^2 \, dist(\Delta, \Lambda^c)} \, \|f\|_\infty \tag{2.3}$$

The free hamiltonian (of the interpolating free measure) is

$$H^0_\beta(\sigma^\Lambda) = -m_\beta \sum_{x \in \Lambda} \sigma(x) \tag{2.4}$$

and the interpolating hamiltonian for $t \in [0, 1]$ is

$$H^+_{\gamma,t}(\sigma^\Lambda) = t H^+_\gamma(\sigma^\Lambda) + (1-t)\, H^0(\sigma^\Lambda) \tag{2.5}$$

Let $Z_{\gamma,\beta,t}(\Lambda, +)$ and $\mu^{\Lambda,+}_{\gamma,\beta,t}$ be the corresponding partition function and Gibbs measure with $+$ b.c. in the sense of (2.2). We then have

$$\log Z^+_\gamma(\beta; \Lambda) = \log Z^0(\beta; \Lambda) - \beta \int_0^1 dt\, E^+_{\gamma,\beta,t,\Lambda}(H^+_\gamma - H^0_\gamma) \tag{2.6}$$

where

$$Z^0(\beta; \Lambda) = \sum_{\sigma^\Lambda} e^{-\beta H^0_\beta(\sigma^\Lambda)} = \cosh(\beta m_\beta)^{|\Lambda|}$$

Setting

$$P^0(\beta) = \frac{\cosh(\beta m_\beta)}{\beta} \tag{2.7}$$

we get, after taking the thermodynamic limit

$$P_\gamma(\beta) = P^0(\beta) + \sum_{y \neq 0} J_\gamma(0, y) \int_0^1 dt\, E^+_{\gamma, \beta, t}(\tfrac{1}{2}\sigma(0)\,\sigma(y) - \sigma(0)\, m_\beta) \tag{2.8}$$

where $\mu^+_{\gamma, \beta, t}$ is the $+$ Gibbs measure with hamiltonian (2.5) and $E^+_{\gamma, \beta, t}$ its expectation. They are well defined as the analogue of Theorem 2.1 holds for the system with hamiltonian (2.5) uniformly in $t \in [0, 1]$. The main point in the proof of such a statement is that the excess free energy functional $\mathscr{F}_{\beta, t}(m)$ associated to $H_{\gamma, t}$ in the Lebowitz–Penrose limit, is

$$\mathscr{F}_{\beta, t}(m) = \int dr [f_\beta(m(r)) - f(m_\beta)] + \frac{\beta t}{4} \iint dr\, dr'\, J(|r - r'|)[m(r) - m(r')]^2$$

where, see (1.6) and (1.7),

$$f_\beta(m) = -\frac{m^2}{2} - \frac{I(m)}{\beta}$$

is independent of the interpolating parameter [same is true for the mean field equation (1.8)]. The whole analysis in refs. 5 and 3 then applies uniformly in $t \in [0, 1]$, details are omitted.

By (1.6) and (2.7)

$$P^0(\beta) = P^{\mathrm{mf}}(\beta) + \frac{m_\beta^2}{2} \tag{2.9}$$

and (2.8) becomes

$$P_\gamma(\beta) = P^{\mathrm{mf}}(\beta) + \tfrac{1}{2} \sum_{y \neq 0} J_\gamma(0, y) \int_0^1 dt\, E^+_{\gamma, \beta, t}([\sigma(0) - m_\beta][\sigma(y) - m_\beta]) \tag{2.10}$$

We have thus completed the first step of the proof, reducing the asymptotic expansion of the pressure to the asymptotic expansion for the two-body v-correlations.

2.2. A priori Bounds

As already mentioned we consider explicity only the case $\beta > 1$, the case $\beta < 1$ can be recovered by setting $m_\beta = 0$, but its analysis could be done alternatively with simpler methods. We define the v-functions by the formula

$$v^+_{\gamma,\beta,t}(x_1,..., x_n) = E^+_{\gamma,\beta,t}\left(\prod_{i=1}^n [\sigma(x_i) - m_\beta]\right) \tag{2.11}$$

where x_i are distinct sites in $\mathbb{Z}^d$.

Proposition 2.2. There are $\theta > 0$ and c_n, $n \geqslant 1$, so that for all γ, $t \in [0, 1]$, $n \geqslant 1$ and n distinct sites $x_1 \cdots x_n$

$$|v^+_{\gamma,\beta,t}(x_1,..., x_n)| \leqslant c_n \gamma^{\theta n} \tag{2.12}$$

Proof. We will first prove that there are θ, c and a positive so that

$$\mu^+_{\gamma,\beta,t}\left(\left|\sum_x J_\gamma(0, x)[\sigma(x) - m_\beta]\right| \geqslant \gamma^\theta\right) \leqslant c \exp\{-a\gamma^{2\theta - d/2}\} \tag{2.13}$$

To prove (2.13) we call $\mathscr{D}_\gamma$ a partition of $\mathbb{Z}^d$ into cubes C_γ of side $\gamma^{-1/2}$, $C_{\gamma,x}$ denoting the cube that contains x. We then set

$$I_\gamma(x, y) = \frac{1}{|C_\gamma|^2} \sum_{x' \in C_{\gamma,x}} \sum_{y' \in C_{\gamma,y}} J_\gamma(x', y') \tag{2.14}$$

and have

$$|I_\gamma(x, y) - J_\gamma(x, y)| \leqslant c \mathbf{1}_{|x-y| \leqslant \gamma^{-1}} \gamma^{1/2} \tag{2.15}$$

Then for $\theta < 1/2$ and γ small enough,

$$\left\{\left|\sum_x J_\gamma(0, x)[\sigma(x) - m_\beta]\right| \geqslant \gamma^\theta\right\} \subset \bigcup_{|y| \leqslant \gamma^{-1}} \left\{\left|\sum_x I_\gamma(y, x)[\sigma(x) - m_\beta]\right| \geqslant \frac{\gamma^\theta}{2}\right\}$$

We omit the proof anologous to the Peierls estimates in ref. 5 that

$$\mu^{+}_{\gamma,\beta,t}\left(\left|\frac{1}{|C_\gamma|}\sum_{y\in C_\gamma}[\sigma(y)-m_\beta]\right|\geqslant\frac{\gamma^\theta}{2}\right)\leqslant c\exp\{-a\gamma^{2\theta-d/2}\} \tag{2.16}$$

for suitable constants c and $a>0$. (2.13) follows from (2.16).

To explain our strategy for concluding the proof of the proposition, let us first consider the easy case where $|x_i-x_j|>\gamma^{-1}$ for all $i\neq j$. By using repeatedly the DLR equations we then get

$$v^{+}_{\gamma,\beta,t}(x_1,..., x_n)=E^{+}_{\gamma,\beta,t}\left(\prod_{i=1}^{n}[\tanh\{\beta J_\gamma(x_i,\cdot)\circ\sigma\}-m_\beta]\right) \tag{2.17}$$

where

$$J_\gamma(x,\cdot)\circ\sigma=\sum_{y\neq x}J_\gamma(x,y)\,\sigma(y) \tag{2.18}$$

Recalling the mean field equation (1.8) we then derive (2.12) from (2.13).

In the general case, i.e., when x_i and x_j are possibly close, we have an expression more complex than (2.17). We need some extra notation.

- Let $x\in\mathbb{Z}^d$ and $f(\sigma)$ a bounded function. Let $\delta_{x,\pm}f(\sigma)$ be equal to the value of f on the configuration obtained from σ by setting $\sigma(x)=\pm 1$. We also define $\partial_x f=\delta_{x,+}f-\delta_{x,-}f$.
- We define $\mathscr{A}^{\pm}_x$ as

$$\mathscr{A}^{+}_x f(\sigma)=[\tanh\{\beta J_\gamma(x,\cdot)\circ\sigma\}-m_\beta]\,\delta_{x,+}f(\sigma) \tag{2.19}$$

$$\mathscr{A}^{-}_x f(\sigma)=\tfrac{1}{2}[1+m_\beta][1-\tanh\{\beta J_\gamma(x,\cdot)\circ\sigma\}]\,\partial_x f(\sigma) \tag{2.20}$$

We will next derive

$$v^{+}_{\gamma,\beta,t}(x_1,..., x_n)=\sum_{\varepsilon_1=\pm 1}\cdots\sum_{\varepsilon_{n-1}=\pm 1}E^{+}_{\gamma,\beta,t}\times\left(\prod_{i=1}^{n-1}\mathscr{A}^{\varepsilon_i}_{x_i}[\tanh\{\beta J_\gamma(x_n,\cdot)\circ\sigma\}-m_\beta]\right) \tag{2.21}$$

that will be obtained proving by induction for $j=1,..., n-1$ that

$$v^{+}_{\gamma,\beta,t}(x_1,..., x_n)=\sum_{\varepsilon_j=\pm 1}\cdots\sum_{\varepsilon_{n-1}=\pm 1}E^{+}_{\gamma,\beta,t}\times\left(\prod_{i=1}^{j-1}(\sigma(x_i)-m_\beta)\prod_{i=j}^{n-1}\mathscr{A}^{\varepsilon_i}_{x_i}[\tanh\{\beta J_\gamma(x_n,\cdot)\circ\sigma\}-m_\beta]\right) \tag{2.22}$$

(2.22) is evidently true when $j=n-1$. We suppose by induction that (2.22) holds for $j=\ell>1$ and prove it next for $j=\ell-1$. We shorthand

$$f(\sigma)=\sum_{\varepsilon_\ell=\pm1}\cdots\sum_{\varepsilon_{n-1}=\pm1}\prod_{i=\ell}^{n-1}\mathscr{A}_{x_i}^{\varepsilon_i}[\tanh\{\beta J_\gamma(x_n,\cdot)\circ\sigma\}-m_\beta]$$
$$\times\prod_{i=1}^{\ell-2}(\sigma(x_i)-m_\beta)$$

and $x\equiv x_{\ell-1}$. Using the identity

$$f(\sigma)=\delta_{x,+}f(\sigma)-\frac{1-\sigma(x)}{2}\partial_x f(\sigma)$$

and the induction assumption, we have

$$v^+_{\gamma,\beta,t}(x_1,\ldots,x_n)=E^+_{\gamma,\beta,t}\left([\sigma(x)-m_\beta]\left\{\delta_{x,+}f(\sigma)-\frac{1-\sigma(x)}{2}\partial_x f(\sigma)\right\}\right)$$

Calling $\alpha=\beta\sum_y J_\gamma(x,y)\,\sigma(y)$ and using the DLR equations

$$v^+_{\gamma,\beta,t}(x_1,\ldots,x_n)=E^+_{\gamma,\beta,t}\Bigg([\tanh(\alpha)-m_\beta]\,\delta_{x,+}f(\sigma)$$
$$+\frac{1+m_\beta}{2}[1-\tanh(\alpha)]\,\partial_x f(\sigma)\Bigg)$$

which proves (2.22) with $j=\ell-1$ (2.21) is proved.

A sort of chain rule holds for a derivative ∂_{x_i} acting on a product of functions. The easiest way to see this is to write

$$\partial_x f(\sigma)=\int_{-1}^{1}d\sigma(x)\,\frac{d}{d\sigma(x)}\,f(\sigma) \tag{2.23}$$

Then $v^+_{\gamma,\beta,t}(x_1,\ldots,x_n)$ can be written as integrals (coming from (2.23)) of a sum of finitely many terms (their number depending on n). Each term is the expectation of a product of n functions, each one being a high order derivative (with respect to variables $\sigma(x_i)$) either of the function $(\tanh(\cdot)-m_\beta)$ or of the function

$$\tfrac{1}{2}(1+m_\beta)(1-\tanh\{\beta J_\gamma(x,\cdot)\circ\sigma\})$$

Each derivative brings in a factor $J_\gamma(x_i, x_j)$ which is bounded proportionally to γ^d. In the product there are functions $(\tanh(\cdot)-m_\beta)$ which are bounded proportionally to γ^θ with probability exponentially close to 1. The number of derivatives and factors $(\tanh(\cdot)-m_\beta)$ is not less than n. Proposition 2.2 is proved. ∎

2.3. BBGKY Hierarchy

We will next derive a BBGKY hyerarchy of equations which is verified by the v-functions and then use it to prove bounds on the v-functions. Let $\mathbb{Z}^{dn}_{\neq}$ be the set of elements in $\mathbb{Z}^{dn}$ with all distinct sites. Let $X=(x_1,..., x_{n-1}) \in \mathbb{Z}^{d(n-1)}$, $x \notin X$ and

$$k_\gamma(x, y \mid X) = \frac{\beta t}{\cosh^2(\beta m_\beta)} J_\gamma(x, y)\, \mathbf{1}_{y \notin X} \tag{2.24}$$

We are going to prove that for any n, X, x as above and $N>1$ there are coefficients $a_\gamma(X, x, Y; N)$ and $R_\gamma(X, x; N)$ so that

$$\begin{aligned} v^+_{\gamma, \beta, t}(x_1,..., x_{n-1}, x) &= \sum_y k_\gamma(x, y \mid X)\, v^+_{\gamma, \beta, t}(x_1,..., x_{n-1}, y) \\ &\quad + \sum_{k=0}^{n+N} \sum_{Y \in \mathbb{Z}^{kd}_{\neq}} a_\gamma(X, x, Y; N)\, v^+_{\gamma, \beta, t}(Y) + R_\gamma(X, x; N) \end{aligned} \tag{2.25}$$

We will also prove that there are absolute constants $c_{n, N, k}$ and $c_{n, N}$ so that

$$\sum_{Y \in \mathbb{Z}^{kd}_{\neq}} |a_\gamma(X, x, Y; N)| \leqslant c_{n, N, k}\, \gamma^{d[(n+1-k)/2] + d\mathbf{1}_{n=k}}, \qquad k \leqslant n \tag{2.26}$$

where $[\,\cdot\,]$ is the integer part and $\mathbf{1}_{n=k}$ is 1 when $n=k$ and 0 otherwise;

$$\sum_{Y \in \mathbb{Z}^{kd}_{\neq}} |a_\gamma(X, x, Y; N)| \leqslant c_{n, N, k}, \qquad k>n \tag{2.27}$$

$$|R_\gamma(X, x; N)| \leqslant c_{n, N}\, \gamma^{\theta N} \tag{2.28}$$

(2.25) is therefore an identity between v functions, with $a_\gamma(X, x, Y; N)$ which are known coefficients and $R_\gamma(X, x; N)$ an "unknown" remainder term which is however "negligibly small."

We will only outline below the proof of the above statements giving the algorithm for computing the coefficients $a_\gamma(X, x, Y; N)$ without doing it explicitly (but in the last subsection we will determine their leading contribution as $\gamma \to 0$). By using the DLR equations we have

$$v_{\gamma, \beta, t}(x_1,..., x_{n-1}, x) = E_{\gamma, \beta, t}\left([\tanh(\beta J_\gamma(x, \cdot) \circ \sigma) - m_\beta] \prod_{i=1}^{n-1} [\sigma(x_i) - m_\beta] \right) \tag{2.29}$$

having dropped the superscript $+$. By (1.8) and recalling that J_γ is normalized, see (1.2),

$$\tanh(\beta J_\gamma(x, \cdot) \circ \sigma) - m_\beta = \tanh(\beta m_\beta + \beta J_\gamma(x, \cdot) \circ [\sigma - m_\beta]) - \tanh(\beta m_\beta) \tag{2.30}$$

We Taylor-expand w.r.t. the variable $\beta J_\gamma(x, \cdot) \circ [\sigma - m_\beta]$ up to order N. The remainder term is $R_\gamma(X, x; N)$. (2.28) then follows from Proposition 2.2.

The first order term of the expansion is

$$\sum_y k_\gamma(x, y \mid X)\, v_{\gamma, \beta, t}(x_1,..., x_{n-1}, y) + \sum_{i=1}^{n-1} J_\gamma(x, x_i)\, E_{\gamma, \beta, t}\left([\sigma(x_i) - m_\beta]^2 \prod_{j \neq x_i = 1}^{n-1} [\sigma(x_j) - m_\beta] \right) \tag{2.31}$$

the first addendum coincides with the first term on the r.h.s. of (2.25). After writing

$$[\sigma(x) - m_\beta]^2 \equiv \frac{1 + \sigma(x)}{2} [1 - m_\beta]^2 + \frac{1 - \sigma(x)}{2} [-1 - m_\beta]^2 \tag{2.32}$$

we add and subtract m_β to $\sigma(x)$ and the expression thus obtained is inserted in (2.31) giving terms that are v-functions multiplied by coefficients which satisfy the bound (2.26). By an analogous procedure we can prove that all the other terms of the Taylor expansion satisfy the bounds (2.26) (2.27). We omit the details and give (2.25)–(2.28) for proved.

The kernel $k_\gamma(x, y \mid X)$, see (2.24), has norm

$$\sum_y k_\gamma(x, y \mid X) \leqslant \frac{\beta t}{\cosh^2(\beta m_\beta)} < 1 \tag{2.33}$$

by (1.8). The resolvent $[1-k_\gamma(\cdot,\ \cdot \mid X)]^{-1}$ is then well defined (as a Neumann series) and we denote by $g_\gamma(x, y \mid X)$ its kernel. We then have

$$v_{\gamma,\beta,t}(x_1,..., x_{n-1}, x) = \sum_z g_\gamma(x, z \mid X) \left\{ \sum_{k=0}^{n+N} \sum_{Y \in \mathbb{Z}^k_{\neq}} a_\gamma(X, z, Y; N)\, v_{\gamma,\beta,t}(Y) + R_\gamma(X, z; N) \right\} \tag{2.34}$$

Let

$$\|v_{\gamma,\beta,t}\|_n = \sup_{X \in \mathbb{Z}^n_{\neq}} |v_{\gamma,\beta,t}(X)| \tag{2.35}$$

calling n the order of the "seminorm" $\|\cdot\|_n$. We have:

Theorem 2.3. There are constants c_n, $n \geqslant 1$, so that for all γ, $t \in [0, 1]$ and $n \geqslant 1$

$$\|v_{\gamma,\beta,t}\|_n \leqslant c_n \gamma^{d[(n+1)/2]} \tag{2.36}$$

Proof. To prove (2.36) we observe that using (2.26), (2.27) and (2.28), (2.34) gives

$$\|v_{\gamma,\beta,t}\|_n \leqslant \sum_{k=0}^{n} C_{n,k,N} \gamma^{d[(n+1-k)/2] + d\mathbf{1}_{n=k}} \|v_{\gamma,\beta,t}\|_k + \sum_{k=n+1}^{N} C_{n,k,N} \|v_{\gamma,\beta,t}\|_k + C_{n,N} \gamma^{\theta N} \tag{2.37}$$

with $C_{n,k,N}$ and $C_{n,N}$ suitable constants.

We iterate N times (2.37) getting

$$\|v_{\gamma,\beta,t}\|_n \leqslant C'_{n,0} \gamma^{d[(n+1)/2]} + \sum_{k=1}^{n+N-1} C'_{n,k,N} \gamma^{d[(n+N-k)/2]} \|v_{\gamma,\beta,t}\|_k + \sum_{k=n+N}^{N^2} C'_{n,k,N} \|v_{\gamma,\beta,t}\|_k + C'_{n,N} \gamma^{\theta N} \tag{2.38}$$

with $C'_{n,k,N}$ and $C'_{n,N}$ suitable constants.

To prove (2.38) we observe that by (2.37) the coefficients multiplying a seminorm of order $k<n$ have a factor $\gamma^{d(n-k)/2}$ if $n-k$ is even and

$\gamma^{d(n+1-k)/2}$ otherwise. The gaining factor is γ^d if $n=k$ while there is no gaining factor if $k>n$. Then after some combinatorics we derive (2.38), details are omitted.

We choose N so that

$$N\theta > dn \tag{2.39}$$

Then, using Proposition 2.2, if $1 \leqslant k \leqslant n+N-1$,

$$\gamma^{d[(n+N-k)/2]} \|v_{\gamma,\beta,t}\|_k \leqslant \gamma^{d[(n+N-k)/2]} c_k \gamma^{\theta k} \leqslant c_k \gamma^{N\theta/2}$$

while for $k \geqslant n+N$ we just observe that

$$\|v_{\gamma,\beta,t}\|_k \leqslant c_k \gamma^{\theta N}$$

Then, by (2.38), Theorem 2.3 is proved. ∎

2.4. Asymptotic Expansion

We are now able to compute $v_{\gamma,\beta,t}(X)$, $X \in \mathbb{Z}^{dn}_{\neq}$, up to order γ^{dM}, for any $M>0$, by "simply" doing the following. We iterate N times (2.34) with $\theta N > dM$. We get a huge number of terms. A first group is made of terms containing R_γ as a factor; a second group has terms with still a v function. What left is what we call $b_{\gamma,\beta,t}(X,N)$. As in the proof of Theorem 2.3, all terms except $b_{\gamma,\beta,t}(X,N)$ are bounded proportionally to $\gamma^{\theta N} < \gamma^{dM}$, thus $b_{\gamma,\beta,t}(X,N)$ and $v_{\gamma,\beta,t}(X)$ are the same up to order γ^{dM}. We then conclude that for any $n \geqslant 1$ and any $X \in \mathbb{Z}^{dn}_{\neq}$ there are coefficients $c_{\gamma,\beta,t}(\ell, X)$ so that

$$v_{\gamma,\beta,t}(X) = \sum_{\ell=[(n+1)/2]}^{M} c_{\gamma,\beta,t}(\ell, X) + o(\gamma^{dM}), \quad |c_{\gamma,\beta,t}(\ell, X)| \leqslant c_{\ell,n}\gamma^{d\ell} \tag{2.40}$$

where $o(\gamma^{dM})$ goes to 0 faster than γ^{dM} and the terms $c_{\ell,n}$ are positive coefficients. The terms $c_{\gamma,\beta,t}(\ell, X)$ can be computed explicitly in terms of the kernels $g_\gamma(\cdot,\ \cdot \mid X)$, here we give the leading ones. For $X=(x_1,x_2)$ we have

$$c_{\gamma,\beta,t}(1, X) = g_\gamma(x_1, x_2 \mid X)[1-m_\beta^2] \tag{2.41}$$

Recall that g_γ is the kernel of $(1-k_\gamma)^{-1}$ and by (2.24):

$$g_\gamma(x_1, x_2 \mid X) = \frac{\beta t}{\cosh^2(\beta m_\beta)} J_\gamma(x_1, x_2) + \cdots \tag{2.42}$$

which is therefore bounded proportionally to γ^d.

For $X=(x_1,..., x_n)$, n even,

$$c_{\gamma,\beta,t}(n/2, X)=\sum_{\substack{\text{all partition of } X\\ \{X_j\}_{j=1,...,n/2};\ X_j\in\mathbb{Z}^{2d}}}\left[\prod_{j=1}^{n/2} c_{\gamma,\beta,t}(1, X_j)\right] \tag{2.43}$$

which is the expression for the nth moment of a centered Gaussian field indexed by $x\in\mathbb{Z}_d$ with covariance $c_{\gamma,\beta,t}(1, X)$.

We conclude obseving that the coefficients $p_n(\beta,\gamma)$ of the expansion of the pressure in powers of γ^d, see (1.10), are obtained by using the expansion (2.40) in (2.10):

$$p_n(\beta,\gamma)=\tfrac{1}{2}\sum_{y\neq 0} J_\gamma(0, y)\int_0^1 dt[c_{\gamma,\beta,t}(n, (0, y))\,\gamma^{-dn}] \tag{2.44}$$

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